Oxford Mathematics

UNIVERSITY OF OXFORD
Mathematical Institute

Guide to Prelims Courses

Contents

1 Introduction to University Mathematics

This eight lecture course runs on consecutive days at the start of term, and introduces the precise mathematical language which will be essential for rigorously developing algebra and analysis in the first year. There is also some discussion of the foundations of mathematics, the base upon which all mathematics ultimately rests: this is a subtle topic though and will not be treated fully till optional courses in the third year. Of more immediate relevance will be guidance on how to try and solve problems in pure mathematics, especially the type typically encountered in algebra and analysis in the first year.

2 Introduction to Complex Numbers

This two-lecture course will introduce complex numbers from scratch, and rapidly develop the theory that you will use through the rest of Prelims. For some of you, much of the material will be familiar from school or college, but no prior knowledge is assumed. Even if you have already studied complex numbers, the course will still be a useful recap and opportunity to make sure we are all using the same notation and terminology.

Complex numbers arise when we introduce a new 'number', called i, with the property that $i^2 = -1$. We consider numbers of the form $a + bi$, where a and b are real numbers. Using complex numbers obviously allows us to solve the equation $x^2 + 1 = 0$, less obviously allows us to solve any quadratic equation with real (or indeed complex) coefficients, and not at all obviously allows us to solve any polynomial equation of any degree with complex coefficients – this is the deep Fundamental Theorem of Algebra, which we'll state but not prove in this course.

Representing complex numbers in the form $a + bi$ gives a natural way to generalise the one-dimensional real line to a two-dimensional complex plane, called the Argand diagram. The real line forms the horizontal axis, and the vertical axis has the purely imaginary numbers (multiples of i); the complex number $a + bi$ corresponds to the point with Cartesian coordinates (a, b). This geometrical interpretation of complex numbers is enormously valuable: we can understand complex numbers using geometry, and we can understand geometry using complex numbers.

Sometimes Cartesian coordinates are not so convenient, and it is more useful to use polar coordinates, where instead of recording the distances from the axes, we instead record the distance from the origin ('radius') and angle above the positive horizontal axis ('argument'). We'll explore this representation, and see the connection with the important result $e^{i\theta} = cos\theta + isin\theta$. One special case of this is the famous equation $e^{i\pi} = -1$.

Complex numbers are a fundamental tool in every mathematician's toolbox, and you'll use them often right from the start of Prelims, so it is well worth trying to become familiar and comfortable with them at the start of your degree.

3 Linear Algebra I

You probably already have one or more strategies for solving a system of two simultaneous linear equations in two variables, such as 3x + 5y = 1 and 2x - 7y = 10. What if you had a hundred linear equations, in a hundred variables? Would your strategies generalise to larger systems of simultaneous linear equations? In this course, you'll meet the idea of a matrix, which is a fundamental idea in mathematics. A matrix is simply a rectangular array of numbers, but importantly we can work with matrices in powerful ways: for example, we can add and multiply matrices. We can record a system of simultaneous linear equations using matrices, and we'll look at an efficient strategy for solving a matrix equation of this sort. We'll see that matrices can also be used to describe certain geometrical transformations in two or three dimensions, such as reflections, rotations and skews.

This generalises to higher dimensions, using the abstract idea of a vector space. A vector space is a mathematical object consisting of a set of 'vectors' and a background set of 'scalars', together with a notion of addition (we can add two vectors) and a notion of scalar multiplication (we can multiply a vector by a scalar). Drawing on our experience of vectors in two or three dimensions, we'll specify a list of axioms that must be satisfied in a vector space: the operations of addition and scalar multiplication must behave nicely. We'll go on to explore ideas such as a basis of a vector space, and develop the theory far enough to define the dimension of a vector space.

We'll explore the deep connection between matrices and vector spaces. We'll define a linear map, which is a nicely behaved map between two vector spaces, and see that linear maps can be represented by matrices.

Linear algebra is of vital importance in mathematics. Unsurprisingly, Linear Algebra II in Prelims, and Part A Linear Algebra, will build on this Linear Algebra I course, but you'll also find yourself using the ideas in many of your other courses, not least in Statistics and Data Analysis later in Prelims. In addition, the style of thinking, often called abstract algebra, that we develop in Linear Algebra I will lead nicely into the Groups and Group Actions course later in Prelims.

4 Linear Algebra II

Linear maps of a vector space to itself are ubiquitous in mathematics and its applications. It is thus important to develop techniques that make it easier for us to understand and visualise such maps. Our first tool will be the determinant which gives us a criterion of whether a linear map is invertible. Then we will be asking what directions and which lines are preserved under the map, and more ambitiously whether we can find a basis so that the associated matrix of the linear map is diagonal. In this context we will prove a version of the Important Spectral Theorem that says that any real symmetric square matrix can be diagonalised via an orthogonal change of basis. As an application we will be able to classify quadratic surfaces in 3-dimensional Euclidean space.

This course is a continuation of Linear Algebra I and will foreshadow much of what will be discussed in more detail in the Linear Algebra course in Part A. We will also revisit some concepts seen in Geometry.

5 Groups and Group Actions

We will introduce from first principles a new mathematical object, a group. Indirectly we have already encountered several examples such as the set of all invertible square matrices or the set of all permutations of a given set. The crucial properties here are that we can compose two elements and that inverses exist. We will explore the abstract and axiomatic set-up and deduce general principles and structure theorems. Ultimately the importance of groups comes from their actions on geometric objects and other mathematical sets (as for example the roots of a polynomial). In the second half of the course we will derive the very useful orbit-stabilizer formula and apply it to counting arguments.

This course starts at the 'beginning' though when we discuss examples we will draw on results from previous courses, especially Linear Algebra. A group is one of the more basic mathematical structures which has proved useful and ubiquitous in modern mathematics as well as in its applications whether it is describing the symmetries of a physical system or that of a crystal. This course is one of the core courses.

6 Analysis I Sequences and Series

The idea of a limit is a fundamental one in maths. Precise definitions of limits, in different circumstances, underpin the area traditionally known as (mathematical) analysis. The main aim of this course is to define what it means for a sequence of real numbers to converge to a limit, and the related notion of a convergent series or infinite sum of real numbers. It was not until around two centuries ago that mathematicians succeeded in finding these definitions, which simultaneously fit with our intuition and lead to rigorous proofs of the properties we expect limits and infinite sums to have.

7 Analysis II Continuity and Differentiability

In Prelims Analysis II, we continue to develop the fundamental theory of the real analysis, which is the very core that all further analysis topics will build on, such as metric spaces, topology, complex analysis, Lebesgue's integration, PDEs, stochastic analysis and etc. We will develop in a rigorous way two concepts of limits and derivatives for real valued functions of one variable. Many familiar basic properties about continuous functions and derivatives are proved by using definitions. The core part of the course is then to establish the most important theorems about continuous functions on intervals, including Intermediate Value Theorem which says a continuous function takes a connected subset to a connected subset. As its significant application, we study monotonic functions on intervals in details, and prove that the continuity of inverse functions of monotone continuous functions. We establish a theorem which says a continuous function maps a compact subset into a compact subset, and proves that a continuous function on a compact subset must be uniformly continuous. These results will be further developed in your future topics such as Topology, Functional Analysis and etc.

In the second part, we establish the theory of differentiability for functions of one variable, and we employ the tool of derivatives to the study of functions with a very simple idea of "looking at" special points on the graphs of differentiable functions. We establish the major and powerful tool in analysis, the Mean Value Theorem. We demonstrate its substantial applications in deriving other analysis tools such the L'Hopital rule, Identity Theorem, Taylor's Theorem and many other useful tools. The results developed in this part will be your major analysis techniques in your study of other topics involving limits and derivatives, which are absolutely fundamental mathematics you have to learn if you want to apply mathematics to solve scientific problems, or you want to study mathematics by its own interest.

8 Analysis III Integration

At school, one learns how to integrate various simple functions such as polynomials ($x^2, x^3 + x$), trigonometric functions (sin x) and some rational functions ($1/(1+x^2)$). In this course, we will take a rigorous look at what it means to integrate a function. Can all functions be integrated? If not, which ones can?

We then turn to a rigorous discussion of the "fundamental theorem of calculus", the idea that integration and differentiation are mutually inverse operations.

Along the way we will establish basic properties of fundamental functions in mathematics such as the exponential and logarithm.

9 Introductory Calculus

There are various skills that every mathematician needs in their 'calculus toolkit' and this course aims to ensure that, whatever your background in calculus, you are equipped with the necessary tools for the courses that you'll meet later on in the academic year. Introductory Calculus starts by looking at techniques for integration and solving ordinary differential equations, and then moves on to look at partial differentiation, which is the differentiation of functions of more than one variable. This then leads on to various methods and applications that may be familiar in the context of one variable (such as finding maxima and minima of functions) but which are now viewed in the more general setting of functions of many variables.

This course gives you the calculus skills necessary for several applied maths courses taken in HT and TT of the first year, including Dynamics, Fourier Series and PDEs, and Multivariable Calculus.

10 Probability

Probability theory is one of the fastest growing areas of mathematics. Probabilistic methods are used in a tremendous range of applications, from genetics to communication theory, from epidemiology to physics to finance. Probability underpins statistics, machine learning, and data science. Probability theory can equally be seen as a part of pure mathematics, with a particularly close relationship to mathematical analysis.

In this course well develop a mathematical approach to probability theory. Well explore important concepts such as independence and conditional probability. Well meet a range of probability distributions and a wide range of probabilistic tools, and develop applications including random walks and branching processes. Well prove some fundamental results about the behaviour of random processes, such as the Law of Large Numbers.

11 Statistics and Data Analysis

If you were asked how to estimate the probability that a particular coin would land heads side up, you would probably be able to estimate that without thinking much about statistical theory. Most would flip the coin N times and count the number of heads (X) and consider that X/N was a good estimate of the probability. This is straightforward. But how accurate, or precise, is this estimate? Are you confident that the probability isnt 0.5?

The course will also cover simple linear regression, demonstrating how a linear relationship between two variables (we could call them X and Y) can be quantified. Expanding on this multiple linear regression simultaneously quantifies the relationship between multiple X variables and a single outcome variable Y. Finally, we will give an overview of the field of unsupervised learning (finding simplifying structure in data sets involving many variables) with real-world examples.

Extracting information from data is a key quantitative skill in so many areas - from scientific discovery, to political science, to understanding consumer behaviour.

12 Geometry

Many of you will be familiar with the use of vectors in geometry, and perhaps also some of the vector algebra associated with the scalar and vector products. This vector algebra itself needs mastering, how to manipulate it and elicit information, but a subtler focus of the course is the validity and consistency of using vectors and co-ordinates at all.

The use of co-ordinates is ubiquitous in mathematics, but are geometric theorems proved using one set of co-ordinates the same as those proved with another set? This directs our attention to notions of isometries and orthogonal matrices. As the course moves on to parametrized surfaces the same question arises when we change parameters for example, is surface area invariant under such a change?

13 Dynamics

How do we describe the physical world around us, and how it changes with time, using mathematics? In this course we study Newton's laws of motion, which describe the dynamics of an object in terms of the forces acting on it. The success of Newton's theory hinges on the fact that a small number of very simple force laws, combined with his laws of motion, very accurately describe an enormous array of dynamical problems. For example, in these lectures we will look at phenomena as diverse as the motion of bodies through fluids, charged particles moving in electromagnetic fields and the motion of rigid bodies under gravity. A highlight of the course uses the law of universal gravitation to account for the motion of planets around the Sun.

More complex descriptions of the physical world may be found in the Part A options on Quantum Theory and Special Relativity, in which many concepts of Newton's theory are modified. Some features appear to be fundamental though: the laws of conservation of energy, momentum and angular momentum developed in this course are in some sense universal, and pervade all of theoretical physics.

14 Constructive Mathematics

The history of mathematics is rife with stories of deep and abstract ideas which eventually found important practical applications after an incubation period lasting decades (or centuries). The catalyst which usually facilitates the percolation of theory to application is the presence of efficient algorithms — if in addition to rigorously establishing the existence of a solution to some system of equations, you are also able to explicitly construct or approximate it, then chances are that you will have made it that much easier for researchers in other fields to make good use of it.

This course serves as an introduction to some of the most famous and useful algorithms across pure and applied mathematics. These have played, and continue to play, a central role in solving many important practical problems. This course begins with the Euclidean algorithm for computing highest common factors of integers, which is the main tool for modern data decryption; and it culminates with Newton's method for numerically solving nonlinear equations via gradient descent, which forms the backbone of contemporary machine learning.

15 Multivariable Calculus

Multivariable calculus sits at an intersection of geometry, analysis, topology and physics. Two main theorems of the course are Stokes' theorem and the divergence theorem. From a pure stance, these can be thought of as two- and three-dimensional versions of the fundamental theorem of calculus. But they were theorems first recognised by physicists and are just as important to a fluid dynamics or electromagnetism course.

The course introduces differential operators div, grad and curl. These have physical significance in themselves, but some questions about them lead to topological answers. Gravity is a conservative force, because the potential energy gained by a particle depends only on its displacement and not the path taken; this means it has zero curl, but whether zero curl means a force is conservative depends on topology of a region.

16 Fourier Series and PDEs

While developing the theory of heat conduction in the early 19th century, Jean-Baptiste Joseph Fourier kick-started a mathematical revolution by claiming that every" real-valued function defined on a finite interval could be expanded as an infinite series of elementary trigonometric functions cosines and sines. The need for rigorous mathematical analysis to assess this astonishing claim led to a surprisingly large proportion of the material covered in prelims, part A and beyond (e.g. the definition of a function, the ε-δ definition of limit, the theory of convergence of sequences and series of functions, Lebesgue integration and Cantor's set theory). The implications of Fourier's claim for practical applications were no less powerful or far-ranging: the decomposition led to deep and fundamental insights into numerous physical phenomena (e.g. mass and heat transport, vibrations of elastic media, acoustics and quantum mechanics) and continue to be exploited today in numerous fields (e.g. signal processing, approximation theory and control theory).

In this course we introduce fundamental results for the pointwise convergence of Fourier's infinite trigonometric series Fourier series. We then follow in Fourier's footsteps by using them to construct solutions to fundamental problems involving the heat equation, the wave equation and Laplace's equation the three most ubiquitous partial differential equations in mathematics, science and engineering.

17 Computational Mathematics

A hallmark of modern scientific and mathematical efforts is the increasing role played by computer programming. From generating conjectures in number theory to numerically approximating solutions to partial differential equations, computers have become an indispensable ally in various aspects of research and development. This course serves as a first introduction to computer programming, with particular emphasis on scientific and numerical computation. It teaches the following skills (among others): finding the best data structures, performing operations on vectors and matrices, numerically solving complicated nonlinear equations, visualizing said solutions, generating and testing conjectures, and implementing algorithms efficiently. This term we will be using the MATLAB software in all the lectures and assignments. As a pleasant byproduct of taking the course, students will also gain expertise in MATLAB.