Oxford Mathematics

UNIVERSITY OF OXFORD

Mathematical Institute

Guide to Part A Courses

Contents

1 Core Courses

1.1 A0 Linear Algebra

This course is a continuation of Linear Algebra I and II in the first year. Around the first 9 lectures will focus on the question: given a linear map on a vector space over a field, can one find a basis so that it is represented by a "nice" matrix? Ideally "nice" would mean diagonal, but we already saw in Prelims that this is not always possible. The main theorems in this part of the course are one on triangular form, the primary decomposition theory, and the Jordan form theorem. On the way towards them we shall study polynomials with coefficients in a field, which leads us to the abstract notions of rings and ideals. The proof of the Jordan form theorem is a rather delicate business and requires a detailed preliminary study of another new construction, that of a quotient vector space.

Having completed our study of nice forms for matrices, we move on to look at dual spaces. In contrast with the first part of the course, the motivation for studying these seems less clear at first and the theory apparently more obtuse. But a little perseverance reveals through them a beautiful symmetry in linear algebra, which allows one to see old ideas in a new light and suggests different approaches to proving theorems.

In the final part of the course we focus on vector spaces over the real or complex field, and bring lengths into play; that is, we look at inner product spaces. We study linear maps on such spaces that behave in particular ways with respect to the inner product, namely orthogonal and unitary matrices, and also self-adjoint matrices which we encountered in the first year in the guise of symmetric matrices. Indeed much of what we shall do here is re-work in a more sophisticated manner the ideas which went into the proof of the spectral theorem for real symmetric matrices in Prelims.

1.2 A1 Differential Equations I

When we solve a differential equation using a certain method, how do we know that we have found all solutions? If we cannot find a solution, can we at least learn something about its properties? This course considers a range of ordinary and partial differential equations that play an important role in applied mathematics and develops techniques to answer these types of questions.

1.3 A2 Metric Spaces and Complex Analysis

One of the fundamental achievements of nineteenth century mathematics was to set the calculus of Newton and Leibniz on firm footing by providing a rigorous definition of a limit. In Prelims you saw how this could then be used to build a reasonably complete theory of analysis in a single real variable. In this course we expand on that theory in a number of ways: Firstly, simply by examining carefully what ingredients were really needed in the definition of a limit, we generalize the theory of continuity substantially by introducing with the concept of a metric space, which is simply a set with a notion of distance, and use it to gain a deeper understanding of basic theorems such as the intermediate value theorem and the extreme value theorem.

Secondly, we investigate the theory of complex differentiability. While at first sight, this appears to be just like the real-variable theory – the product rule, quotient rule and chain rule etc. all hold with almost identical proofs – it rapidly reveals itself to be very different, due to the richer geometry of the complex plane and the tighter connection between differentiability and integrability. From the point of view of real analysis it is a fool's paradise – any function which is differentiable is automatically infinitely differentiable for example!

After developing the fundamental features of the theory, Cauchy's theorem, the integral formulae, the identity principle and the residue theorem, we show how powerful this new tool-kit is by applying it to sum series such as $\sum_{n=1}^{\infty} 1/n^2$ and evaluate a wide range of definite integrals.

2 Long Options

2.1 A3 Rings and Modules

Historically there have been many wrong proofs of Fermat's Last Theorem, and commonly those proofs were faulty for assuming nice algebraic properties (e.g. unique factorization) of sets amongst the complex numbers. So the study of rings, as abstract algebraic structures, partly grew out of understanding factorization which is a large part of the A3 course. A further aspect of the course is the idea of creating larger fields over which a polynomial may factorize - this introduces much of the groundwork necessary to the Part B Galois Theory course.

The second half of the course is a study of linear algebra over rings - that is, a study of modules. Much of the familar linear algebra no longer applies - modules don't generally have bases - but some familiar techniques like row and column operations still apply. The culmination of the course is in the Structure Theorem. This has two immediate corollaries - the classification theorem for finite abelian groups and the Jordan normal form for complex matrices. That two such diverse results follow from the theorem speaks to the power of the abstract technology.

The A3 course is essential to all Part B algebra: Galois Theory, Representation Theory, Commutative Algebra.

2.2 A4 Integration

Riemann integration is fine if you are only interested in integrating functions which are bounded and continuous "almost everywhere" over bounded intervals. It is possible to extend Riemann integration to some extent by allowing "improper" integrals, but there is very little theory for such integrals and they are still confined to functions which are continuous almost everywhere.

Henri Lebesgue's doctoral thesis (1902) introduced a new integration theory which far extends the scope of Riemann integration. The construction of the Lebesgue integral assigns a notion of length to all (reasonable) subsets of $\mathbb{R}$, and approximates the integrand by taking horizontal strips across its graph. Continuity requirements are replaced by a notion of "measurability" which includes all (reasonable) functions, and integrals over unbounded intervals or of unbounded functions are fully included in the theory. The end-products include abundant results about passing limits through integrals, integrating series term-byterm, passing derivatives through integrals, and interchanging double integrals, as well as constructions of spaces of integrable functions which are complete in the sense of Cauchy sequences converging. The theory extends to many contexts other than classical integration.

2.3 A5 Topology

Convergence and continuity are fundamental notions that one studies in 1st year Analysis and develops later in Metric Spaces. These notions are based on our spatial intuition of 'closeness' and Topology provides an abstract setting for them, giving a common language to both Analysis and Geometry. Abstraction brings simplification and the ability to use our spatial intuition to visualize and prove theorems about mathematical objects that do not live in space at all. For example one uses Topology to study spaces of functions and prove existence theorems for Differential Equations or one equips with a Topology the set of prime ideals of a ring.

One may see in Topology one of the traits of modern mathematics: that of seeking analogies between different mathematical objects and attempting to use notions and ideas developed in a field to an entirely different setting.

The last part of the course is closer to Geometry, studying connectedness, giving a way to 'cut and paste' spaces to create new ones and elucidating what it means for two spaces to be actually 'the same (i.e. 'homeomorphic'). We give an introduction to Manifolds, the main object of study of modern Algebraic Topology and Geometry, by studying surfaces in detail. By the end of the course it will be clear why Topologists dont distinguish doughnuts from coffee cups.

Topology leads to several Analysis and Geometry part B/C courses. For example on the Analysis side the part B Functional Analysis I,II courses and on the Geometry side the part B Topology and Groups and Geometry of surfaces and several part C courses building on those.

2.4 A6 Differential Equations 2

The study of ordinary differential equations (ODEs) underpins much of applied mathematics, including topics in theoretical mechanics, mathematical physics and mathematical biology, for example. Much of this course concerns *linear boundary value problems*. For example, consider the following apparently simple second-order ODEs:

$$y''(x) + y(x) = 1, \qquad (1a) \qquad\qquad y''(x) + \pi^2 y(x) = 1, \qquad (1c)$$

$$y''(x) + y(x) = \tan x, \qquad (1b) \qquad\qquad y''(x) + \pi^2 y(x) = 1 - 2x, \qquad (1d)$$

each to be solved on the interval $0 < x < 1$ and subject to the boundary conditions $y(0) = 0 = y(1)$.

Equation $(1a)$ can easily be solved by spotting that $y(x) = 1$ is a particular integral, but there is no obvious way to spot a particular integral for equation $(1b)$! The general solutions of equations $(1c)$ and $(1d)$ can both be found using elementary methods, but one soon finds that $(1c)$ has *no solution* satisfying the given boundary conditions, while $(1c)$ has an *infinite number* of such solutions! These observations prompt the following questions about ODEs similar to (1).

1. Is there a general method to construct the solution, without resorting to guesswork?

2. Given suitable boundary conditions, when does a solution exist? When is it unique?

2.5 A7 Numerical Analysis

Given finitely many function values of an otherwise unknown function, how should we predict its function value at a new input point? How can we numerically compute the integral of a function? What should we consider to be the solution to an overdetermined linear system of equations, and how can we compute it? How would we compute the eigenvalues and eigenvectors of a large matrix on a computer? How can such computations be made reliable, fast and accurate? What approximation and complexity (execution-time) guarantees can we give for our algorithms?

These and other related questions are answered in this course via an approach that combines mathematical rigour (to analyse approximation guarantees) with clever algorithmic design (to reduce complexity). Anyone interested in solving mathematical models on a computer, be they from continuous or discrete mathematics and from any application area, will find this course an indispensable introduction to scientific computing and algorithmic design.

2.6 A8 Probability

This course builds on the Prelims course in Probability. Beyond the classical limit theorems (Laws of Large Numbers and Central Limit Theorem), a substantial part of the course is devoted to stochastic processes such as Markov chains, which generalise random walks. Whether a frog jumping on lily pads, a nearest-neighbour walk on a graph, a population size, or a gamblers wealth, there is an abundance of examples, including those of practical use in genuine applications in finance, insurance and the sciences, where the next state of a system is determined by the previous state and some extra randomness. Such stochastic processes are called Markov chains. Problems that we address either in general or in examples include hitting probabilities, return times, periodicity, stationarity, limiting behaviour. In particular, we will explain why, as Kakutani put it, a drunk man will find his way home, but a drunk bird may get lost forever.

This course leads on to several Part B courses, most directly to the Applied Probability course. Most Statistics options rely on Probability, as well as on Part A Statistics. The course on Probability, Measure and Martingales uses ideas from the Part A Integration course to explore further classes of stochastic processes, also leading to further courses in Stochastic Analysis and Mathematical Models of Financial Derivatives. Part A Probability is also a prerequisite for Stochastic Modelling of Biological Processes and useful for courses on Information Theory and on Actuarial Science that are currently offered as Part B courses.

2.7 A9 Statistics

In simple situations we might regard the observations in a dataset as being the observed values of i.i.d. random variables: what can we infer from the dataset about the underlying distribution of these random variables? This course focuses on parametric models, where the underlying distribution depends on a finite-dimensional parameter. Early in the course we look at a way to assess if a certain parametric model is plausible, e.g. to answer the question: is it reasonable to assume a set of data is normally distributed. We consider estimating the value of a parameter, providing a point estimate or a confidence interval, and by developing the theory of hypothesis testing we also see how we can "test" whether or not a parameter takes a particular value and what is meant by a "test". These questions about estimation, confidence intervals and testing are also considered from the "Bayesian" point of view in the last third of the course, linking to Bayes' Theorem from Prelims Probability, where probabilities are updated to conditional probabilities as information (data) becomes available.

Part A Probability is recommended for this course, but is not essential. If doing this course without Part A Probability you will need to be familiar with Prelims Probability and a couple of lectures' worth of Part A Probability. The material in this course is relevant for most future statistics courses (including machine learning, computational statistics, statistical lifetime models), and for statistics courses in future years it is also strongly advisable to take Part A Probability.

2.8 A10 Fluids and Waves

What can your visits to the hair salon teach you about the Method of Images? What is the difference between sitting on a river bank and watching a punt go by, as opposed to travelling in the punt?

This course introduces students to the mathematical theory of inviscid fluids. The theory provides insight into physical phenomena such as complex potentials, flight, vortex motion, and water waves. The course also explains important concepts such as conservation laws and dispersive waves and. It therefore serves as an introduction to the mathematical modelling of viscous flows.

2.9 A11 Quantum Theory

When we look at very small scales, e.g. atomic scales, the laws of classical mechanics (the ones learnt in Dynamics) break down, and are replaced by a different set of 'quantum rules'. The position of particles is not a well-defined concept any longer and cats can be both, dead and alive. Quantum Mechanics is the most important physical framework of the last century and with a huge range of applications. In this course we introduce their rules, and learn how to work with them in some simple examples.

3 Short Options

3.1 ASO Number Theory

Take two odd primes p and q. Suppose that there is a square number that is p more than a multiple of q. Is there also a square number that is q more than a multiple of p? How does this answer depend on the primes p and q? This question is answered by the beautiful theorem of quadratic reciprocity. The importance of this theorem is demonstrated by the amount of time that Gauss spent thinking about it: he published at least six proofs of the theorem himself. In this course, you'll meet the theorem and a proof, having developed the relevant and very elegant theory first.

In addition to the mathematical attractions of this subject material, there are also practical applications to cryptography, and the course will spend some time looking at these applications as well as developing the underlying theory.

This course develops in detail the ideas of modular arithmetic and Fermat's Little Theorem which were briefly discussed during the Prelims Groups and Group Actions course, and in turn leads on to Part B Algebraic Number Theory and other related courses.

3.2 ASO Group Theory

Groups are best thought of as symmetries of geometric objects. As such they play an important role in large parts of mathematics and its applications. In this course we will study their internal structures and start to classify them: How many groups of a given size are there? Given two groups how can we combine them to form a new group? What are the primary building blocks of groups?

This course will build on your knowledge from the Prelims course Groups and Group Actions. We will break down groups via their normal groups and quotient groups, and study their subgroups of prime powers. It will prepare you for several third year courses. Both B3.1 Galois Theory and B3.5 Topology and Groups will lead directly on from this.

3.3 ASO Projective Geometry

Ever wondered whether the parallel lines in the plane meet at infinity or not? Come to this course to find out in Projective Geometry, they most certainly do! Projective geometry is a version of geometry which includes, but is subtly different from, our ordinary (Euclidean) notion of geometry; it can be thought of as the geometry of perspective drawing. It possesses a pleasant non-Euclidean symmetry called duality, which turns points into lines and vice versa, and leads to most theorems appearing in attractive pairs.

The course builds on linear algebra, giving an illustration of how ideas of that algebraic subject can be applied in a different context. It develops the notion that there is not just one notion of "geometry". Other geometries will be introduced in later courses in relativity theory and B3.2 Geometry of Surfaces, while non-linear projective geometry will be developed in the courses B3.3 Algebraic Curves and more generally in C3.4 Algebraic Geometry.

3.4 ASO Introduction to Manifolds

The notion of a curved geometry is fundamental to our modern understanding of the mathematics and physics of space (and space-time) around us. This course gives the first glimpse of the technical underpinnings of this subject, by developing a geometrically meaningful notion of differentiation of functions of several variables. The theory leads to a general notion of invertible functions, as well as a formalisation of the tangent space to a curved geometry.

3.5 ASO Integral Transforms

How do we describe a point mass, force, charge, or heat source mathematically? We can define the cumulative distribution function of a discrete random variable, but what is the density function? Can the derivative of a step function make sense? All these questions are handled in the beautiful and elegant framework of distributions or generalised functions. This is the first topic of the course.

The second topic is a generalisation of some of the Fourier series ideas from Prelims. We can represent a function defined on a finite interval as a weighted sum of basis functions a Fourier series but what if the interval is infinite? The sum becomes an integral and we study two versions, called the Laplace and Fourier transforms. We see how to use them to solve problems in differential equations, and we see how they interact with the distributions introduced in the first part of the course.

This short option is directly relevant to a wide range of later courses across the whole spectrum of mathematics, from analysis and partial differential equations to probability and a host of physical application areas.

3.6 ASO Calculus of Variations

We all know that the shortest distance between two points is a straight line but how do you prove this, and what happens if the points are on a curved surface? What is the shape of a curve of a fixed prescribed length that maximises the area under the curve? Problems like these are known as variational problems, and occur in many areas of mathematics. For example, the motion of the simple pendulum can be framed as a variational problem.

The aim of this course is to formulate questions of the above form in terms of variational problems and then derive a system of differential equations, known as the Euler-Lagrange equations, to determine the solution.

3.7 ASO Graph Theory

Networks are everywhere in our world: transportation networks (e.g. roads), communication networks (e.g. phones), social networks (e.g. Facebook). What kind of mathematical questions arise when we think about networks? One such question that gripped the popular imagination in the mid 20th century, and was the subject of a famous experiment by Stanley Milgram, was the "small world problem", also known as "six degrees of separation": can any two people be linked by a chain of people, of length at most six, such that any link in the chain is a pair of people who know each other?

This course develops the theory of graphs (the mathematical abstraction of networks) through the lens of various practical problems: minimum cost spanning trees (making cheap connections), shortest paths (getting from A to B), bipartite matching (assigning jobs to contractors) and the Chinese Postman Problem (delivering items). The solutions of these problems, and the proofs that these are correct, require various elegant mathematical ideas - indeed, the great variety of proof techniques in Graph Theory is at once a challenge in the beginning and a reward at the end. These ideas in turn are fundamental to more advanced courses, such as B8.5 Graph Theory, C8.3 Combinatorics and C8.4 Probabilistic Combinatorics within the Oxford Mathematics curriculum, and more generally to any further study of networks from a theoretical or practical viewpoint.

3.8 ASO Special Relativity

In 1905, Albert Einstein revolutionized humanitys understanding of space and time when he published his seminal paper, On the Electrodynamics of Moving Bodies. In a stroke, Einstein showed that the apparent invariance of the speed of light under changes in reference frame could be reconciled with the philosophical (if not the technical) principle of relativity by radically altering ones view on the relationship between space and time. The consequences of this discovery have been far-reaching, including the development of atomic energy and a complete reformulation of the theory of gravity in the form of Einsteins later theory of general relativity.

Starting from basic postulates, we derive the structure of a unified, relativistic space-time known as Minkowski space. We address various apparent paradoxes (the twin paradox and barn door paradox) and their dissolution in an appropriate relativistic treatment. Elementary considerations of how to generalize the rules of dynamics to a relativistic setting leads to the famous relation between the mass and energy of an object, E=mc2.

This course builds on first-year algebra, geometry, and dynamics, and in turn leads on to Part C General Relativity and many MMathPhys courses such as Quantum Field

Theory and String Theory.

3.9 ASO Mathematical Modelling in Biology

Consider two populations, say rabbits and foxes. How do these populations evolve over time? We will explore different modelling frameworks including discrete and continuous-time models. Specifically we will focus on the predator-prey models describing the dynamics of rabbits and foxes. In this course we will introduce different mathematical models and mathematical techniques to analyse these systems.

This course builds on ideas of differential equations and in turn leads on to Part B Further Mathematical Biology as other modelling and differential equation courses.