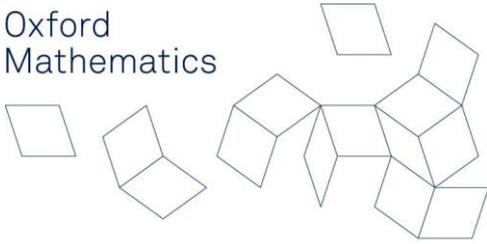


Oxford
Mathematics



Guide to Options at Part C

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1 C1.1 Model Theory

Term - Michaelmas Mathematical logic revolves around a duality between sentences or structures for a fixed language. For a sentence ψ , we can look at the class of models of ψ conversely to a structure N we associate complete theory $Th(N) := \{\psi : N \models \psi\}$. Model theory adapts this duality to the study of mathematical structures.

In previous mathematics classes, you have encountered a variety of universally axiomatized theories such as the theories of groups, rings, or vector spaces over some field. On the other hand you have seen specific structures, such as the field of real numbers.

How to bridge the gap between them? How much does the complete theory tell us about a structure? How do you ever find the complete theory of an infinite structure? Given a universally axiomatized theory, is there any canonical way to describe a complete theory containing it?

These apparently different questions are deeply intertwined, and we will learn some methods for answering them. We will concentrate on the special case of categorical structures, that are completely determined by the cardinality of their universe and their theory. They will provide us with many examples where we can prove quantifier elimination and thereby also address the gap between a universal and a complete theory. Countably categorical structures will be characterised, with an interesting relation to group theory. The methods involved - omitting and realising types, atomic and saturated models - will illustrate techniques valid in greater generality.

2 C1.2 Gödel's Incompleteness Theorems

Term - Hilary How much can we know through pure logical deduction? This course looks in detail at Gödel's famous answer, which is, "Not much". The First Incompleteness Theorem says, roughly, that any usable system of axioms for arithmetic (by which is meant the structure of the natural numbers, with addition and multiplication), is incomplete in that there is a truth, call it G , about the natural numbers which cannot be proved from that system of axioms. The method used to establish the First Incompleteness Theorem is almost as famous as the statement itself (largely because of the many excellent popular books on the topic), and proceeds by finding a way in which the formal language of arithmetic can talk about itself. The statement G turns out, in this framework, to mean "This statement cannot be proved" (and is a variant of the Liar Paradox). It is obvious that G can only be proved if it is false; assuming that our logic is sound, G then can't be proved, so is true.

The First Incompleteness Theorem is bad enough, and is fatal to any program to discover all of the truths of arithmetic (as was once hoped, by Hilbert and others). But even if we can't know everything, surely we can expect to be certain about what we do know? Well, not according to the Second Incompleteness Theorem. We exploit the ability of the language of arithmetic to talk about itself to write down a formula Con_S , where S is a system of axioms for arithmetic, which means " S is consistent". Surely the question of whether S is consistent is one we want to know the answer to before we even consider using S . But again, we find ourselves disappointed. The Second Incompleteness Theorem says, approximately, that if S is a usable system of axioms for arithmetic, then Con_S cannot be proved from S . This has the upshot that if we do want a proof that the system S is consistent, we can only get one in a more powerful system—which is also more likely itself to be inconsistent.

This course contains other theorems with a similar flavour. We also begin to explore provability logic, and to examine in more detail exact what a system of axioms S can and cannot do.

It will be assumed that you are familiar with propositional logic and first order predicate logic, and with the Completeness Theorem (also proved by Gödel). Some of the theorems in this course are similar to theorems of computability theory, and there is similarity of atmosphere with the Axiomatic Set Theory course (because the major theorem in that course, about the consistency of the Axiom of Choice, was also proved by Gödel using some similar techniques). The Second Incompleteness Theorem is very important in the further development of set theory.

3 C1.3 Analytic Topology

Term - Michaelmas

Compactness is a powerful property. If you know that a space X is compact, and even more if it is Hausdorff as well, you know a great deal about X , and a great deal about continuous functions from X to other spaces. (Just think of such elementary theorems as that a continuous function on a closed bounded interval is bounded and attains its bounds.) So, what conditions ensure that a space X is compact and Hausdorff? If it isn't, are there other properties that might do equally well, for certain purposes? Or can we embed X in a space in another, bigger space that is compact and Hausdorff (or, in the technical language, perform a compactification)?

Highlights of this course include Tychonoff's Theorem, which says that a product of any number of compact spaces is compact, the construction of the Stone-Čech compactification, and the proof that all metric spaces possess a weaker (but still very useful) property called paracompactness.

We also touch on topics such as connectness, weakenings and strengthenings of Hausdorffness, metrisability (the question of when a topological space is possibly a metric space), and the theory of Stone duality, which enables one to convert questions about logic into questions about topology and vice versa. It will be assumed that you have studied enough topology to know the definition of an abstract topological space and of a continuous function, and be familiar with such concepts as Hausdorffness and compactness, and basic theorems concerning these. Some knowledge of set theory, and in particular of the Axiom of Choice, would be helpful.

4 C1.4 Axiomatic Set Theory

Term - Hilary

This course is one of the most formal mathematics courses you are likely to take and hence an informal description is probably not very helpful and might in fact be misleading. However, we take a similar approach in the course to make the formalism intuitive and thus no further harm should result from an informal description.

Some mathematicians are worried about using the Axiom of Choice and try to avoid it for good and not-so-good reasons. In this course we develop the theory necessary to show that adding the Axiom of Choice to the other usual axioms of set theory (ZF) does not introduce any new inconsistencies. We then use the same machinery to show that adding the Continuum Hypothesis does also not introduce any new inconsistencies.

Although non-examinable and only if time allows, we sketch the method of Forcing and how to use it to show that the negation of the Continuum Hypothesis is likewise consistent with ZFC.

The major challenge in this course is to accept that we can study set theory as a mathematical theory (in a very weak meta-theory) and the consequences this entails, namely that some of the usual notions of mathematics (in particular the powerset of an infinite set) are not in fact as well-defined as we wish to believe. Although seemingly abstract, the tools we develop to address this challenge are suitable (and used!) in a wide field of mathematics e.g. some results in finite combinatorics are most easily proven by relying on the axiom of choice - we show that in principle these proofs can be turned into proofs without the axiom of choice.

The relevant background is a first course in set theory (e.g. Part B Set Theory) showing how to use the axioms of set theory to build a foundation of most of mathematics. It is also important to be comfortable in formal first order logic, although no specific results will be used.

5 C2.1 Lie Algebras

Term - Michaelmas

Lie algebras are objects which appear in many areas of mathematics. Their origin is in the study of topological transformation groups, which we now call Lie groups. It turns out that two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic. Moreover there is the Lie correspondence where closed, connected subgroups (normal subgroups) of the Lie group correspond to subalgebras (ideals) of the associated Lie algebra.

This course will introduce Lie algebras over the complex numbers and study them using linear algebra and some basic representation theory. In the first part of the course we will study nilpotent and solvable Lie algebras and prove the theorems of Engel and Lie. Then we will proceed to the study of an important class of subalgebras, the Cartan subalgebra. Finally we prove the classification of semisimple Lie algebras using certain geometric/combinatorial structures in Euclidean space, called root systems. These are finite sets of vectors with lots of symmetries. It turns out that each simple Lie algebra is completely described by a very small graph (in fact a tree) associated to its root system, called the Dynkin diagram which encodes the angles between the vectors of the root system.

6 C2.2 Homological Algebra

Term - Michaelmas

In what ways does the category of modules over a ring R behave like the category of vector spaces over a field? In many aspects they behave in the same ways: they are both abelian categories.

But, unlike in the case of vector spaces, not every submodule $A \subseteq B$ is a direct summand; it might instead fit in a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, where C is the quotient of B by A .

For fixed R -modules A and C , how many possible B 's can one insert in the middle? These are the sort of questions that homological algebra tries to answer.

This course builds on Part A Rings and Modules; Introduction to Representation Theory B2.1 is recommended but not essential.

7 C2.3 Representation Theory of Semisimple Lie Algebras

Term - Hilary

This course builds on "Introduction to representation theory" (part B) and "Lie algebras" (part C) to give a classification of the finite dimensional representations of a semisimple Lie algebra such as $\mathfrak{sl}(n)$ (the Lie algebra of matrices with trace zero). The finite dimensional representations are constructed as quotients of natural infinite dimensional representations (Verma modules). The methods and results covered in this course are fundamental tools in modern representation theory and have applications to geometry, mathematical physics, combinatorics and number theory.

8 C2.4 Infinite Groups

Term - Michaelmas

It already became apparent in the first courses in Prelims, and it was further reinforced by more advanced courses in Parts A and B, that groups are interesting especially when seen as groups of transformations of certain spaces, preserving certain structures that those space might be endowed with (whether linear, geometric, number theoretic or other). From this point of view, infinite finitely generated groups are among the most interesting, as they are studied in connection with (and allow a better understanding of) key structures in Geometry, Topology, Analysis and Number Theory.

The course begins with a brief presentation of free groups, in some sense the “largest” among finitely generated groups, and a necessary preliminary to any study of infinite groups.

It then focuses on three generalizations of the notion of abelian groups, that is (in increasing order of generality) the nilpotent, polycyclic and solvable groups. While each of these classes is introduced using a clear cut algebraic definition, the methods to investigate them are diverse, from algebraic (e.g. measuring to what extent they can be approximated by finite groups, via the property of residual finiteness; or whether they can be represented as groups of matrices), to geometric and algorithmic methods.

The course ends with two key results: that polycyclic groups can be characterized among solvable groups by the fact that they can be represented as groups of matrices with integer entries; and that nilpotent groups can be characterized among solvable groups by their polynomial growth (the Milnor -Wolf theorem).

Knowledge of the first and second-year algebra courses is helpful but not mandatory. The course relates to the part C Geometric Group theory course.

9 C2.5 Non-Commutative Rings

Term - Hilary

Non-commutative rings are too general class of rings to have meaningful theory. Therefore we need to impose some extra finiteness conditions. In this course we will study rings with either the descending or the ascending chain condition. These are called the Artinian and Noetherian rings respectively.

This course builds on techniques from the Representation theory and Commutative algebra from part B. They are not essential but highly recommended and the students will be expected to be familiar with some of their results.

The first part of the course deals with Artinian rings. We prove a classification of primitive Artinian rings which generalizes the Artin-Wedderburn theorem about semi-simple algebras in Representation theory. We then proceed to study the Ore condition and non-commutative localization. The final part of the course is focused on developing dimension theory of filtered Noetherian rings. This is a generalization of the concept of Krull dimension in commutative algebra. Finally we prove Bernsteins inequality and Gabbers theorem on the integrability of the characteristic variety.

10 C2.6 Introduction to Schemes

Term - Hilary

The theory of schemes was developed in the late 1950s by A. Grothendieck and his school, in an attempt to give an intrinsic description of the objects of algebraic geometry, as opposed to the classical extrinsic description in terms of varieties, which always come with an embedding into affine or projective space.

The language of schemes is a lot more flexible and natural than the language of varieties and it is a good setting for the study of infinitesimal phenomena, which are difficult to grasp in the classical language. Another strong point of the language of schemes is that it provides a unified description of arithmetic and geometric objects. For example, commutative rings and Riemann surfaces can both be viewed as schemes. Schemes have played an essential role in the solutions of some of the most important conjectures in number theory, e.g. in the proof of the Weil conjectures (which concern the number of solutions of equations over finite fields) by P. Deligne and the Mordell conjecture (which predicts that curves of genus at least two over a number field have finitely many rational points) by G. Faltings.

The language of schemes is now standard and is used in all the research articles in algebraic geometry.

11 C2.7 Category Theory

Term - Michaelmas

It's not an accident that the kernel of a linear map between vector spaces is called by the same name as the kernel of a group homomorphism, even though they are not identical concepts. There are close similarities between the proofs of the rank-nullity formula and the first isomorphism theorem for groups, though they are different theorems. There are many such ideas and arguments which appear in different areas of mathematics (especially algebra, but also in topology and elsewhere), and category theory attempts to bring them together.

A category has objects and morphisms between them satisfying some natural conditions; for example, there is a category of vector spaces over a field k (with linear maps between them), and a category of topological spaces (with continuous maps between them). We can also consider functors between categories; for example there are forgetful functors from each of these categories to the category of sets, as well as inclusion functors from the categories of abelian groups to the category of groups and from the category of compact Hausdorff spaces to the category of topological spaces. There are also functors which associate to any set S the free group generated by S , or the vector space with basis S , or the discrete (or indiscrete) topology on S .

If B is a basis for a vector space V then linear maps from V to another vector space W correspond naturally with maps of sets from B to W . Homomorphisms from a group G to an abelian group A correspond naturally with homomorphisms (of abelian groups) to A from the quotient $G/[G,G]$ of G by its commutator subgroup $[G,G]$. Continuous maps from a topological space X to a compact Hausdorff space Y correspond naturally with continuous maps to Y from the so-called Stone-Cech compactification of X . In this course you'll see how we can think of all these statements as relating functors between the relevant categories.

This Category Theory course links with other courses such as Algebraic Geometry, Algebraic Topology (which is based on the study of functors from the category of topological spaces to categories of groups and other algebraic structures), Homological Algebra and Representation Theory. However it is not necessary to combine any of these with Category Theory.

12 C3.1 Algebraic Topology

Term - Michaelmas

To follow.

13 C3.2 Geometric Group Theory

Term - Hilary

Which loops on a surface can be contracted to a point? It turns out this translates into an algebraic question: the word problem for the fundamental group of the surface. Somewhat surprisingly Dehn solved this problem at the beginning of last century using hyperbolic geometry.

Since then the study of groups given by finitely many generators and relations has developed in close connection with Topology and Geometry. This course will focus on two strands of this relation: the study of groups via their actions on Trees and the study of the ‘coarse geometry of groups. In both instances geometric tools allow us to shed light on algebraic problems such as the subgroup structure of a group and its word and conjugacy problems.

The field of modern Geometric Group Theory interacts with Geometry/Topology but also with Logic/Model Theory. Some recent results witnessing these connections are the resolution of Thurston’s virtual Haken conjecture in 3-manifolds and Tarski’s problems on the elementary theory of free groups.

This course develops further some ideas seen in the part B Topology and Groups course, builds on the Prelims and Part A Group Theory courses, and is related to the part C Infinite Groups course.

14 C3.3 Differentiable Manifolds

Term - Michaelmas

A (differentiable) manifold X is a geometric space which locally (on small scales) looks like \mathbb{R}^n , that is, in small regions we can parametrize X by local coordinates (x_1, \dots, x_n) , but globally (on large scales) X may have some interesting topology, or shape. For example, the surface of a sphere S^2 and the surface of a doughnut T^2 are both 2-dimensional manifolds, which are topologically distinct. Manifolds are special topological spaces X with some extra information, a smooth structure, which allows us to do calculus – differentiation and integration – on X , which do not make sense just on a topological space. We can also consider geometric structures on X , such as a Riemannian metric g , giving a notion of distance on X . Often one studies geometric structures satisfying partial differential equations, since differentiation makes sense on manifolds.

Manifolds are the basic language in which much of modern physics, physical applied mathematics, and PDEs, is written. For example, Einsteins theory of General Relativity models space-time as a 4-dimensional manifold S with a geometric structure, a Lorentzian metric, satisfying a PDE, Einsteins equations. Cosmology concerns the large-scale shape of S . Manifolds also have everyday applications, for example we can model the surface of the earth as a 2-sphere S^2 with a Riemannian metric g , lengths of paths $\gamma : [0, 1] \rightarrow S^2$ are given by integrating g , and shortest paths between two points are geodesics”, satisfying an ODE.

The course introduces manifolds and smooth maps between them, geometric structures on manifolds including vector bundles, the tangent and cotangent bundles, exterior forms, orientations, and Riemannian metrics, and various kinds of differentiation and integration on manifolds. We study the topology of manifolds via de Rham cohomology.

A good understanding of topological spaces, such as A5 Topology, is essential, and B3.2 Geometry of Surfaces is useful background.

15 C3.4 Algebraic Geometry

Term - Michaelmas

C3.4 together with C3.1 Algebraic Topology and C3.3 Differentiable Manifolds constitute the three key foundational geometry courses in Part C. Their flavour differs slightly due to the class of functions they investigate on a geometrical object: C3.4 studies polynomial/rational functions, C3.1 continuous functions, and C3.3 smooth functions. Their flavour is similar to Commutative Algebra, Homological Algebra, and Analysis, respectively.

This course studies "varieties": spaces which, at least locally, are defined as solution sets of polynomial equations. One tries to describe geometrical properties of these spaces by studying the ideal that these polynomials generate inside the ring of all polynomials. B3.3 Algebraic Curves is a natural precursor, as it describes one-dimensional varieties. The C3.4 course deals with algebraic varieties, projective varieties and quasi-projective varieties. Two highlights are the proof that the study of algebraic varieties is equivalent (in the category theory sense) to studying finitely generated reduced algebras, and the study of quasi-projective varieties is equivalent to studying finitely generated field extensions.

Part A Rings and Modules is essential, and an understanding of basic terminology from Part A Topology is assumed. B2.2 Commutative Algebra is a useful but not essential course, as many of the algebraic notions and results from B2.2 get reformulated in a geometrical way. This MT course ties in naturally with the HT course C2.6 Introduction to Schemes, which is a vast modern generalisation of the classical algebraic geometry studied in C3.4, and it is the natural language in which much of modern geometry and number theory is formulated nowadays. For a more algebra-oriented (rather than geometry-oriented) selection of courses, students could combine C3.4 and C2.6 with the algebra courses C2.2 Homological Algebra and C2.7 Category Theory.

16 C3.5 Lie Groups

Term - Hilary

The group of rotations of n -dimensional space is an infinite (in fact uncountable) group, but also has the geometric structure of a manifold, informally a higher-dimensional generalisation of a smooth surface.

Such objects are called Lie groups—many of the most important examples are matrix groups, but other examples also occur. The theory of Lie groups involves a blend of geometry, topology, analysis and algebra and also is foundational for modern theoretical physics as Lie groups often arise as symmetry groups in gauge theory.

17 C3.7 Elliptic Curves

Term - Hilary

An elliptic curve can be described by an equation of the form $y^2 = x^3 + A * x + B$ where A and B are integers satisfying $4 * A^3 + 27 * B^2 \neq 0$. The set of (projective) rational solutions to such an equation forms an abelian group, called the Mordell-Weil group. The group law is described using an elegant geometric construction which involves intersecting the elliptic curve with straight lines. In this course we study the Mordell-Weil group, proving a number of its key properties and giving an application to integer factorisation.

First we shall show that the subset of points (x, y) in the Mordell-Weil group of finite order is itself a finite subgroup, and moreover such a point must have coordinates x and y in the integers (rather than just the rationals). Proving this will take us into the beautiful world of p -adic numbers, and a careful study of what are called “formal groups”. Our results give an effective method for finding all such points on an elliptic curve. Next we tackle the whole group, and by a technique known as two-descent, show that it is finitely generated. The rank of the free part of the Mordell-Weil group is called the rank of the elliptic curve. Our methods give a way of finding an upper bound on the rank of any given curve, which occasionally one can show equals the rank by finding enough points of infinite order. Finally we conclude with an important and fun application of elliptic curves to integer factorisation due to Lenstra, a problem of great importance in cryptography.

18 C3.8 Analytic Number Theory

Term - Michaelmas

How many primes are there less than 1 billion? More generally, how many primes are there which are less than X , for a given large number X ? A partial answer is given by the Prime Number Theorem, and a more comprehensive answer would be given by the Riemann Hypothesis - one of the most important unsolved problems in mathematics.

In this course we will develop ideas from analysis to study problems in number theory - particularly complicated arithmetic objects such as the prime numbers. This gives an elegant general theory which focuses on the Riemann Zeta function and its connection to prime numbers, leading to a proof of the Prime Number Theorem. We will also see how the Riemann Hypothesis connects to this theory and would give much more precise answers.

This course builds on ideas met in analysis courses (particularly Analysis I, as well as basic complex analysis and Fourier analysis from Part A and B) to study in more detail primes and objects encountered in Part A Number Theory.

19 C3.9 Computational Algebraic Topology

Term - Hilary

To follow.

20 C3.10 Additive and Combinatorial Number Theory

Term - Hilary

This course explores some classic questions in number theory. Starting by proving that every positive integer is a sum of four squares, we then ask about higher powers: is every positive integer the sum of a few 10th powers, for example? How big can a set of integers be if it does not contain three numbers in arithmetic progression (such as 3, 5, 7)? Which finite sets of integers are almost closed under addition? A lot of techniques are introduced in the course, varying from Fourier analysis (developed from scratch) to more combinatorial arguments.

21 C4.1 Further Functional Analysis

Term - Michaelmas

Many of the historical roots of functional analysis lie in trying to solve equations of the form $Tx = y$, where y is a given element of a Banach space Y , x is to be found in a Banach space X and T is a bounded linear operator from X to Y . For instance, we may wish to solve a recurrence relation of the form $p(S)x = y$, where $x, y \in \ell^1$, p is a polynomial and S is the left-shift operator on ℓ^1 . Given y , does this equation have a solution x ? If so, is the solution unique? If there is more than one solution, can we impose constraints on x so that the solution becomes unique?

These questions turn out to have particularly satisfying answers when we are dealing with a so-called *Fredholm operator* (to be defined in the course). With any Fredholm operator we may associate an important integer-valued quantity known as its *index*. For instance, when $T = p(S)$ as above, then T is a Fredholm operator if and only if p has no roots lying on the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and if this condition is satisfied then the index of the operator coincides with the number of times the curve $\theta \mapsto p(e^{i\theta})$, $0 \leq \theta \leq 2\pi$, winds around the origin in the counter-clockwise direction. We thus obtain a non-trivial connection between an analytic quantity (the index) and a topological quantity (the winding number).

One of several major aims of this course will be to develop the theory of Fredholm operators, which in particular will allow you to prove the statements made above. By taking this course you will gain a deeper understanding not only of bounded linear operators between Banach spaces but also of the geometric properties of Banach spaces themselves. The course builds on earlier courses in functional analysis (such as the Oxford courses B4.1 Functional Analysis I and B4.2 Functional Analysis II) but will be slightly more topological in flavour. There are several (loose) connections with other Part C courses such as C1.3 Analytic Topology and C4.3 Functional Analytic Methods for PDEs.

22 C4.3 Functional Analytic Methods for PDEs

Term - Michaelmas

This is a natural continuation from Part A Integration and Part B B4.1-B4.2 Functional Analysis and/or B4.3 Distribution Theory and Fourier Analysis. It begins with a study of the familiar Lebesgue spaces but in a more functional analytic setting, then continues to the study of Sobolev spaces and ends with an application to the study of elliptic partial differential equations. In particular, existence and regularity for solutions of certain elliptic partial differential equations are obtained in some abstract setting without the need of knowing the explicit form of the solutions. No prior knowledge on the analysis of partial differential equations is required, though some minimal familiarity with Part A Differential Equations can be sometimes helpful. The course provides valuable background for the Part

C courses on Calculus of Variations, Fixed Point Methods for Nonlinear PDEs, and Finite Element Methods.

23 C4.6 Fixed Point Methods for Nonlinear PDEs

Term - Hilary

An important question that is not only very relevant in the theory of PDEs, but that should also be considered before trying to model the solution of a PDE, is whether a solution of a given PDE even exists.

Fixed point theory provides a powerful approach that allows us to establish the existence of a solution of an equation in many situations where one cannot actually solve the equation, be this for a non-linear equation on a finite dimensional set, for a differential equation (as seen at the beginning of the part A course DE1), or for PDEs.

In this course we first develop abstract fixed point theory both on finite and infinite dimensional spaces, proving in particular the Theorem of Brouwer that assures that any continuous map from a closed finite dimensional ball to itself has a fixed point, and its infinite dimensional counterpart, the Theorem of Schauder.

We will then combine these abstract fixed point results with the modern approach to PDEs to establish the existence of solutions (living in a suitable Sobolev space) of important classes of PDEs, including the stationary Navier-Stokes equation and the p-Laplace equation.

The style of proofs and exercises in this course is similar to the one of the Part A course in Integration and the Part B courses on Functional Analysis. The course is a natural continuation of the MT course on Functional Analytic Methods for PDEs (the relevant results of which will be recalled in the course) and will give you in particular a chance of applying results and techniques on weak solutions of PDEs and ideas from Functional Analysis in the context of non-linear equations.

24 C4.8 Complex Analysis: Conformal Maps and Geometry

Term - Michaelmas

This course will start where the second year Complex analysis ended: conformal maps. We will start by proving the Riemann mapping theorem that any non-trivial simply connected domain can be conformally mapped to the unit disc. Conformal maps behave nicely inside the domain, but near the boundary they are getting more and more irregular. So the next question that we are going to address is their boundary behavior. In particular, we will see that under mild geometric conditions these maps are continuous up to the boundary. In particular, this allows using conformal maps to solve the Dirichlet boundary problem for the Laplacian.

The main idea is that solutions to the Dirichlet boundary problems are conformally invariant. We will see that they are not the only objects that do not change under conformal transformations and will develop a rather general and very geometric theory of such invariants.

25 C5.1 Solid Mechanics

Term - Michaelmas

In introductory physics, mechanics, and fluids, you have learned about some basic laws and principle such as Hookes law and the conservation of energy. The question addressed in this course is how do we build a general theory for a material? What is the general mathematical framework, and what are the physical assumptions that lead to particular theories such as the classical theory of Newtonian mechanics, the Navier-Stokes or the Navier equations. In previous courses, these questions were introduced for particular systems. In this course the general framework of continuum mechanics based on tensorial fields is introduced. Then, using the basic principles of balance of linear and angular momenta and energy, a general theory is obtained. The different branches of mechanics are obtained by specifying the material response. The second part of the course focuses on the most difficult case: solid mechanics and elasticity. The full nonlinear (exact) theory of nonlinear elasticity will be obtained and particular examples will be used to understand possible behaviours of solids. Finally, a linearisation of the governing equations will lead to the theory of linear elasticity and will open the door for the sequel of this course: Elasticity and Plasticity.

This course should appeal to students interested in physical applied mathematics but also to those with interest in the fundamental and mathematical nature of all physical theories.

26 C5.2 Elasticity and Plasticity

Term - Hilary

Robert Hooke (1678) wrote

“... it is... evident that the rule or law of nature in every springing body is that the force or power thereof to restore itself to its natural position is always proportionate to the distance or space it is removed therefrom, whether it be by rarefaction, or separation of its parts the one from the other, or by condensation, or crowding of those parts nearer together.”

Hooke devised his law while designing clock springs, but noted that it appears to apply to all “*springy bodies whatsoever, whether metal, wood, stones, baked earths, hair, horns, silk, bones, sinews, glass and the like.*” Hooke’s law can be formulated as stating that the *stress* in an elastic body (i.e. the internal force per unit area) is a linear function of the *strain* (i.e. the relative deformation), and this empirical law underpins the theory of *linear elasticity*.

Some solid materials, such as rubber, can withstand large strains, such that the relation between stress and strain eventually becomes nonlinear, as studied in the course C5.1 Solid Mechanics. However, many solids cease to be elastic long before nonlinear effects become important. Under excessive stress, a *brittle* material (e.g. a digestive biscuit) will undergo *fracture*, while a *ductile* material (e.g. a metal paper-clip) will become *plastic* and undergo irreversible deformation.

In this course, we show how the basic model for linear elasticity can be extended to describe real-world phenomena including fracture and plasticity.

27 C5.4 Networks

Term - Hilary

What do the human brain, social systems, the World Wide Web and metabolism have in common? They are all composed of a large number of elements in interaction and exhibit complex dynamical patterns. Network science provides a general framework to represent those systems, by means of nodes and edges, and tools to understand the relations between their structure and dynamics.

This course covers a range of modern research questions in the field, and includes the design of appropriate methods to uncover information in large networks, the construction of random graph models and the study of dynamical systems, such as random walks and epidemic spreading, on networks.

The course requires basic notions of linear algebra, probability and some computational experience. Relevant notions of graph theory and stochastic processes will be reviewed.

28 C5.5 Perturbation Methods

Term - Michaelmas

To follow.

29 C5.6 Applied Complex Variables

Term - Hilary

Complex Analysis has remarkable application in many practical areas of science. This course makes use of complex variables and transforms to solve a range of problems in fluid dynamics, solid mechanics and wave propagation.

The course begins with a brief review of the necessary material from core complex analysis, as is covered in the Part A course Complex Analysis. This includes Cauchy's theorem, contour integration, conformal mapping, and the important fact that the real and imaginary parts of analytic functions satisfy Laplace's equation. It will also assume some basic knowledge of Fourier transforms, as found in the Part A short option Integral Transforms.

We will investigate how to conformally map between many different domains, how to use this to solve steady heat-flow problems, and to solve fluid-flow problems with both fixed boundaries and free surfaces. We'll also use integral expressions of holomorphic functions to solve some problems in the theory of flight and to examine the stress distribution around cracked elastic materials. Finally we generalise the definition of Fourier transforms to solve some mixed boundary value problems, where different conditions apply on different parts of a boundary.

30 C5.7 Topics in Fluid Mechanics

Term - Michaelmas

The course is a direct successor to the two third year fluid mechanics courses, Viscous Flow and Waves and Compressible Flow. It uses the techniques learned there to study an eclectic mixture of fluid mechanical applications. The four topics which currently comprise the syllabus are thin film flows and flow in porous media, convection, rotating flows and two-phase flow.

Thin film flows use the methods of lubrication theory to provide approximate models which are used in a variety of industrial applications, including coating flows and glass drawing, and the analysis of foam drainage. Models of flow in porous media use similar techniques to derive approximate descriptions of groundwater flow.

Convection describes fluid motion driven by buoyancy (light fluid rises) and is of widespread significance in nature, ranging from the motion of the Earth's mantle which drives plate tectonics to the convective plumes seen in volcanic ash clouds or above chimney stacks.

Rotating flows are of most obvious application in meteorology, and this strand will trace the basic mechanism of mid-latitude air flows, and the way in which baroclinic instability leads to the weather patterns which are regularly described in weather forecasts.

Two-phase flows are of ubiquitous interest in nature and industry, ranging from flows in boilers, nuclear cooling systems, pints of Guinness and oil wells to snow avalanches, volcanic eruptions and geysers. Their description involves the extension of the Euler equations to accommodate conservation laws of mass and momentum for each phase, and will be used to study density-wave oscillations which occur in strongly heated flows.

31 C5.9 Mathematical Mechanical Biology

Term - Hilary

To follow.

32 C5.11 Mathematical Geoscience

Term - Michaelmas

Mathematical models provide fundamental understanding of many environmental and geophysical problems. Examples include weather forecasting, flood prediction, volcanism, and climate change. This course demonstrates such use of mathematics by studying a number of different applications in the geosciences. In particular, we will build and analyse models of the Climate, Rivers, and Glaciers.

The models are mostly based on ordinary and partial differential equations, and are solved using techniques that most students have met in their first and second year. Useful mathematical background includes non-dimensionalization, linear stability analysis of ODEs, phase plane analysis, and the method of characteristics for first-order hyperbolic equations. Some knowledge of fluid mechanics is useful though not essential. A certain amount of physics and chemistry will also be introduced. The focus is on application and interpretation of mathematics to a wide and interesting branch of science.

33 C5.12 Mathematical Physiology

Term - Michaelmas

The course is a direct successor to the third year course on Further Mathematical Biology, and addresses many similar issues in a physiological context, including the occurrence of waves and oscillations in a variety of systems, ranging from the intracellular (the action of calcium release) through extracellular (wave propagation in neurons) to organic (the mechanism of the heartbeat and the circulation) and systemic (respiration and blood cell production).

The early applications use similar techniques to the third year course, including phase plane techniques and one-dimensional wave propagation in partial differential equations, but the latter parts introduce new techniques to describe wave propagation in two or three dimensions, and the use of delay-differential equations, which have widespread application in systemic control systems, and which describe certain diseases such as Cheyne–Stokes breathing and chronic myelogenous leukaemia.

34 C6.1 Numerical Linear Algebra

Term - Michaelmas

So important are linear transformations in Mathematics that matrices arise in practical applications of all sorts.

Fundamental matrix decompositions and iterative approximation algorithms form the core of methods for computing relevant information in most situations—such methods of matrix computation rarely use the approaches that we might employ with pencil and paper. Thus this course will describe and develop algorithms for the computation of eigenvalues and singular values and the ubiquitous problem of the solution of linear systems. Knowledge of linear algebra is, of course, necessary as well as an interest in practical constructive (computational) methods.

35 C6.2 Continuous Optimisation

Term - Hilary

The solution of optimal decision-making and engineering design problems in which the objective and constraints are nonlinear functions of potentially (very) many variables is required on an everyday basis in the commercial and academic worlds. A closely-related subject is the solution of nonlinear systems of equations, also referred to as least-squares or data fitting problems that occur in almost every instance where observations or measurements are available for modelling a continuous process or phenomenon, such as in weather forecasting or in machine learning. The mathematical analysis of such optimization problems and of classical and modern methods for their solution are fundamental for understanding existing software and for developing new techniques for practical optimization problems at hand.

36 C6.3 Approximation of Functions

Term - Michaelmas

When it comes to actually working with a function $f(x)$ – evaluating it at particular values, integrating it from a to b , finding its zeros – it is amazing how often mathematicians turn to methods related to polynomials. A function $f(x)$ may be rather abstract, but a polynomial $p(x)$ is very concrete! And so, whether explicitly or implicitly, mathematicians very often first approximate f by p , then evaluate or integrate or find the roots of p .

The foundation of such techniques is the field known as approximation theory. It has deep conceptual aspects, such as the Weierstrass approximation theorem, which was developed to help answer the question, "What is a function?" And it is also the starting point for numerical analysis.

37 C6.4 Finite Element Methods for PDEs

Term - Hilary

To follow.

38 C6.5 Theories of Deep Learning

Term - Michaelmas

Deep learning is a method that currently achieves state of the art results in many learning tasks such as image classification, finding rare events in science problems such as astrophysics, as well as many unusual task such as generating synthetic images that are distributional consistent with a training data set. In this course you will learn the leading architectures, some theoretical results showing the importance of depth in these architectures, methods to train the networks, and theoretical issues on how to initialise the training procedure. Lectures in this course present recent results in this fast moving research area and the course is assessed through a mini-project where you will have a chance to read and speak to research articles discussing one of the various theories about the efficacy of deep learning.

39 C7.4 Introduction to Quantum Information

Term - Hilary

The classical theory of computation usually does not refer to physics.

Pioneers such as Turing, Church, Post and Goedel managed to capture the correct classical theory by intuition alone and, as a result, it is often falsely assumed that its foundations are self-evident and purely abstract. They are not! Computers are physical objects and computation is a physical process. Hence when we improve our knowledge about physical reality, we may also gain new means of improving our knowledge of computation. From this perspective it should not be very surprising that the discovery of quantum mechanics has changed our understanding of the nature of computation. In this series of lectures you will learn how inherently quantum phenomena, such as quantum interference and quantum entanglement, can make information processing more efficient and more secure, even in the presence of noise.

40 C7.5 General Relativity I

Term - Michaelmas

Einstein's general theory of relativity is one of the jewels in the crown of modern physics. It explains how the force of gravity actually comes from the curvature of space-time itself. Matter, such as the sun and Earth, can stretch and warp space-time. In turn, the shape of space-time tells matter how to move through the universe. In this course, you'll learn how to describe these concepts using differential geometry and see how this beautiful theory predicts orbits of planets, large-scale cosmology and the existence of black holes.

This course builds on Part A Special Relativity and Part B Classical Mechanics and Electromagnetism, and leads on to Part C General Relativity II.

41 C7.6 General Relativity II

Term - Hilary

In this second course on general relativity we dive much deeper into its mathematical beauty and its geometric description of gravity. We unravel how Einsteins equations describe the propagation of gravitational waves, which have only recently been experimentally observed, and explore in greater depth the fascinating world of black holes. This course builds on Part C General Relativity I.

42 C8.1 Stochastic Differential Equations

Term - Michaelmas

Stochastic differential equations (SDEs) model quantities that evolve under the influence of noise and random perturbations. They have found many applications in diverse disciplines such as biology, physics, chemistry and the management of risk. Classic well-posedness theory for ordinary differential equations does not apply to SDEs. However, stochastic integration allows to develop a new calculus for such equations (Ito calculus). This leads to many new and interesting insights about quantities that evolve under randomness, that have found many real-world applications. This course is a introduction to stochastic differential equations.

43 C8.2 Stochastic Analysis and PDEs

Term - Hilary

To follow.

44 C8.3 Combinatorics

Term - Michaelmas

To follow.

45 C8.4 Probabilistic Combinatorics

Term - Hilary

To follow.

46 C8.5 Introduction to Schramm-Loewner Evolution

Term - Hilary

There is a wide range of random discrete curves that have (or are conjectured to have) a conformally invariant scaling limit. The simplest example is that of the simple random walk on \mathbb{Z}^2 which converges to Brownian motion on \mathbb{R}^2 . The Brownian motion has an amazing property that its image under a conformal transformation is a (time-changed) Brownian motion.

In 1998 Oded Schramm introduced a one-parameter family of random conformally invariant curves that describe all possible conformally invariant scaling limits of various lattice models in two dimensions that appear in statistical physics. These models are known as the Schramm-Loewner Evolution (SLE).

We will start with a crash course in function theory and briefly introduce several useful notions from complex analysis. After that, we will introduce SLE curves and study their basic properties. This will require a number of tools both from complex analysis and stochastic analysis. Finally, we will discuss the connection between interfaces in the lattice models of statistical physics and SLE curves.

47 C8.6 Limit Theorems and Large Deviations in Probability

Term - Hilary

Various models in statistical physics, biology and quantitative finance, arise as scaling limits of simple interacting particle systems. The course lectures aim to build the fundamental tools for obtaining such probabilistic limiting models starting from simple physical systems, which have universal scientific value and are widely used in stochastic analysis and applications.

Taking a simple random walk which is a simple Markov chain and can be studied without sophisticated mathematics. By taking a scaling limit a continuous model, Brownian motion, may be obtained which in turn makes deep connections with the heat equation and diffusion. Brownian motion arises as the limit for many different underlying random walks and thus is a ‘universal’ object which can be used to study macroscopic phenomena of the original physical system.

The same idea may be applied to more sophisticated models such as the Cooper pairs of electron systems, which leads to non-linear equations such as the Ginzburg-Landau PDE. In this course we develop the basic tools for convergence of probability distributions, mainly various compactness criteria for families of distributions via martingales and Aldous’ stopping times.

Skorokhod’s topology on path spaces is studied in detail. The second part of the course will address the convergence rate of weak convergence, in the context of large deviations, and make connections with the idea of entropy from statistical mechanics.