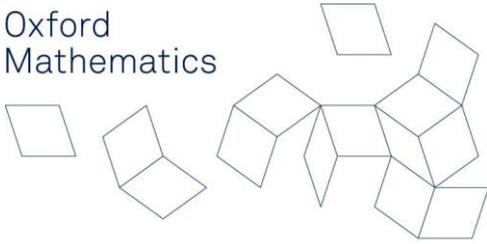


Oxford
Mathematics



Mathematical
Institute

Guide to Options at Part B

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1 B1.1 Logic

Term - Michaelmas

Mathematics is about certainty, exactness, truth and proofs. Ideally one would like to have an algorithmically producible list of axioms for all of mathematics (such a list cannot be finite) together with a (formal) proof system from which one can derive that the axioms together with the proof system are consistent (so they don't lead to contradictions) and complete (so every true mathematical statement is provable within the system). By Gödel's famous Incompleteness Theorem, this dream will never ever come true.

However, if one uses the restricted (though still rather expressive) so-called first-order predicate language together with a suitable proof system (a deductive calculus) one can prove (again Gödel's) Completeness Theorem for that system. This will be the core theorem of the course, along with many of its applications, as well as a general introduction to formal languages and proofs.

While there are no specific prerequisites for the course, some mathematical maturity and acquaintance with algebraic structures will be helpful. The course (or equivalent) will be assumed for the 4th year course on Model Theory and (to some extent) the course on Gödel's Incompleteness Theorem and that on Axiomatic Set Theory. It is well complemented by B1.2 Set Theory. It is recommended to anyone interested in the foundations of mathematics.

2 B1.2 Set Theory

Term - Hilary

Set theory is generally accepted as the "Foundation" of mathematics in that all the objects studied in mathematics can (arguably) be considered as sets whose properties are governed by a small number of intuitive axioms.

This course introduces the mostly widely accepted formulation of such an axiom system, known as Zermelo-Fraenkel set theory.

Set theory is also the setting for studying various properties of infinite sets, in particular two notions of infinite "numbers" introduced by George Cantor, who created the subject in the late nineteenth century: Cardinal numbers and Cantor's surprising discovery that infinite sets come in different "sizes", and ordinal numbers generalizing induction and recursive definitions.

3 B2.1 Introduction to Representation Theory

Term - Michaelmas

While a *group* can be defined abstractly via a set of axioms, in ‘nature’, groups manifest themselves via their actions on various spaces. For example, the dihedral group with 8 elements can be defined abstractly with generators and relations, but we tend to think of it as the group of symmetries of a square. The action of a dihedral group on the plane containing the regular polygon is a 2-dimensional *representation* of this group. More generally, a representation of a group G on a vector space V is an ‘action’ of G on V , i.e., a group homomorphism $G \rightarrow GL(V)$, where $GL(V)$ is the group of invertible transformations of V . For example, whenever one has a group G -action of set X , there are natural G -representations V attached such as the linear space of functions on X with values in a fixed field. The typical questions in representation theory are to determine the G -invariant subspaces of a representation V and to ‘classify’ all the irreducible representations (those which do not have proper nonzero invariant subspaces).

In this course, we begin in a more general setting, that of modules over associative unital algebras (an algebra has both the structure of a ring and a compatible structure of a vector space, think of the ring of all n by n matrices with coefficients in a field, for example). Then we specialise to the setting of *semisimple* associative algebras and we apply the theory to the group algebra of a finite group G . We continue with the concept of complex characters of a finite group which is a clean and beautiful part of the theory. Finally, we present certain applications to the structure of finite groups and connections with algebraic number theory, most notably Burnside’s theorem which says that every group of order $p^a q^b$ (p, q primes) is solvable.

4 B2.2 Commutative Algebra

Term - Hilary

To follow.

5 B3.1 Galois Theory

Term - Michaelmas

Galois Theory is the study of the solutions of polynomial equations from the point of view of group theory. For example, it associates with every polynomial equation over the rational numbers a finite group, called the Galois group, which naturally acts on the roots of the equation by permutations.

This means that the classification of finite groups will have consequences for the qualitative theory of the solutions of polynomial equations.

For example, Galois theory can be used to prove that the solutions of a rational polynomial equation of degree at least five cannot in general be described in terms of its coefficients using only rational functions and extractions of n -th roots (where n is a natural number). This was a classical open problem in the early nineteenth century, when Evariste Galois invented this theory. Another classical problem, which can be solved using this theory, is the problem of the trisection of the angle. Using Galois Theory, it is possible to show that there is no general construction of the third of a given angle using only ruler and compass, thus solving the problem in the negative.

6 B3.2 Geometry of Surfaces

Term - Michaelmas

B3.2 looks at three classes of surfaces: topological surfaces (1), smooth surfaces (2) and Riemann surfaces (3). The choice of class depends on whether we are interested in studying continuous functions, smooth functions or holomorphic functions on geometrical objects.

(1) builds on Part A Topology. We revisit carefully the notions of Euler characteristic and Orientability, and we describe how to modify surfaces by performing attachments. This is a natural precursor to C3.1 Algebraic Topology.

(2) uses the tools from Linear Algebra and Analysis from Prelims/Part A to describe properties of smooth surfaces in R^3 . Concepts such as Tangent Spaces, Curvature, and Geodesics (i.e. shortest paths) are explored. Two highlights are: the Gauss-Bonnet theorem that the Euler characteristic can be recovered from integrating the Gaussian curvature, and secondly that the Euler characteristic equals the number of maxima and minima minus the number of saddle points for almost any given choice of smooth function. This is a natural precursor to C3.3 Differentiable Manifolds and C7.5 General Relativity.

(3) builds on Part A Complex Analysis. One defines the Degree of a holomorphic map between Riemann surfaces, which is a vast generalisation of the fundamental theorem of algebra. We also encounter a non-Euclidean geometry: the hyperbolic plane. A highlight is the Riemann-Hurwitz theorem, which allows one to compute the Euler characteristic of a Riemann surface in terms of simple local information about a holomorphic map to/from a known Riemann surface. This ties in well with the B3.3 course on Algebraic Curves, which includes Riemann surfaces but also allows for spaces to have singularities. Both these courses are precursors of C3.4 Algebraic Geometry and C3.7 Elliptic Curves.

7 B3.3 Algebraic Curves

Term - Hilary

Algebraic curves C in \mathbb{R}^2 are curves given by an equation $p(x, y) = 0$, where $p(x, y)$ is a real polynomial. Some examples are the circle $x^2 + y^2 = 1$, conics, and the cubic $y = x^3 - x$. It is interesting to generalize such curves in two ways, both of which make curves better behaved: we replace \mathbb{R} by \mathbb{C} , which is algebraically closed, and we compactify the complex plane \mathbb{C}^2 to the complex projective plane \mathbb{CP}^2 by adding "points at infinity". We will study the geometry of curves C in \mathbb{CP}^2 given by an equation $P(x, y, z) = 0$, where $P(x, y, z)$ is a complex polynomial which is homogeneous of some degree $d > 0$.

Some examples of results we prove are that if C, D are curves in \mathbb{CP}^2 of degrees m, n which do not contain a common sub-curve $E \subset C, D$ then C, D intersect in mn points, counted with multiplicity. If C is a curve of degree d which is non-singular (has no "bad points") then as a topological space C is a "sphere with g holes", where $g = (d - 1)(d - 2)/2$. We relate algebraic curves in \mathbb{CP}^2 with Riemann surfaces, a kind of geometric space defined using complex analysis and holomorphic functions.

This is a first course in Algebraic Geometry, and is useful background for C3.4.

8 B3.4 Algebraic Number Theory

Term - Hilary

Complex numbers can be thought of as pairs of real numbers with multiplication given by $(a,b)(c,d)=(ac-bd, ad+bc)$. In this course, we will discuss systematic ways of defining field structures on n -tuples of rational numbers. These are called *algebraic number fields*. They have natural subrings given by n -tuples of integers that have refined structures extending well-known structures of integers, in particular, a theory of primes and factorisation. The material of this course can be considered as 'generalised arithmetic' on the one hand, and 'specific ring theory' on the other. As a consequence, it will endeavour to present a harmonious mixture of theory and example, building on previous courses in number theory, rings, fields, and Galois theory.

9 B3.5 Topology and Groups

Term - Michaelmas

This course combines two different topics that are familiar from first and second year courses. It beautifully and powerfully illustrates how topology can be studied via algebra and vice versa. The central theme here is the fundamental group that can be associated to every topological space. It is build out of paths to and from a chosen base point with multiplication corresponding to concatenation of paths and taking inverses to reversing the

direction of the paths. One of our first results will be that any finitely generated group is the fundamental group of some compact space. Furthermore, these fundamental groups determine covering spaces (many to one maps that are locally homeomorphisms) which correspond precisely to the subgroups of the fundamental group. In turn this relation can be used to study the groups themselves. Thus, for example, we will be able to prove that every subgroup of a finitely generated free group is again free.

It will be useful to have familiarity with the content of the second year course Topology. Master level course that build on this course include Algebraic Topology and Geometric Group Theory.

10 B4.1 Functional Analysis I

Term - Michaelmas

Most of the objects with which we work daily in many fields of mathematics such as functions, sequences, vector fields, do not form finite dimensional vector spaces as encountered in Linear Algebra, but rather live in infinite dimensional spaces.

In Functional Analysis I we develop the basic theory of infinite dimensional normed spaces and of continuous linear maps between such spaces. The course is a natural continuation of the Prelims courses in Analysis and the Part A courses in Linear Algebra, Metric Spaces and Integration and at the same time builds the theoretical foundations for many advanced courses.

11 B4.2 Functional Analysis II

Term - Hilary

To follow.

12 B4.3 Distribution Theory and Analysis of PDEs

Term - Michaelmas

To follow.

13 B4.4 Fourier Analysis and PDEs

Term - Hilary

To follow.

14 B5.1 Stochastic Modelling in Biological Processes

Term - Hilary

Imagine that we built a mathematical model of a living cell which can tell us how the cell functions in health and disease. Imagine that our model can predict interactions of the cell with pharmaceutical drugs. Then we could use our model not only to learn the first principles of life, but also to find new medicines and understand disease. In this course, you will learn mathematical and computational methodologies which will help you to become an important part of an interdisciplinary team trying to achieve such an ambitious dream. A living cell contains molecules of many different types and sizes. Some molecular species are present in very low abundances. If there is 1 molecule on average in the cell, we know that at any given time there could be 0, 1 or more molecules, but always an integer number - there will never be half a molecule - and the time evolution of the system will be described as a stochastic process. Thus we will build on your Prelims and Part A Probability courses.

Although some simple stochastic models can be analysed using only a pen and paper approach, one cannot apply stochastic modelling to more complicated examples without using computers. Thus, our discussion is not only built around mathematical equations and their analysis, but also around the corresponding algorithms. You will write your own computer codes in Matlab which you learned in your Prelims Computational Mathematics course. By the end, you might be the only mathematician in your interdisciplinary team. You will replace a complex biological system (which you do not fully understand) by a complex computer code and your team workers, biologists, will rely on that code being correct. Such a computer code can give your team interesting (counter-intuitive) predictions, but you need to be confident that your computer code is doing what it should do.

One option is to break your large code into small subsystems which you fully understand, i.e. small subsystems which you can analytically solve. Course B5.1 studies such simple models and will use methods from a number of your Prelims and Part A courses to analyse the corresponding mathematical equations, including your Prelims Fourier Series and PDEs, Probability, Introductory Calculus, Multivariable Calculus and Computational Mathematics courses, and Part A Differential Equations and Probability courses. For simple models, the same information can be obtained from the analysis of suitable differential equations as from stochastic simulations in Matlab, which will help you to write correct computer codes.

15 B5.2 Applied Partial Differential Equations

Term - Michaelmas

Physical phenomena, when translated to the language of mathematics, very often are expressed in the form of partial differential equations. From problems in traffic flow, to light propagation, to the spot patterns of leopards, to the stock market, to piling sugar on a spoon, to name but a few, partial differential equations are the applied mathematician's primary tool in understanding the world around us.

So how do we analyse and solve them? In previous courses, in particular Fourier series and PDEs in the 1st year, you learned a few techniques. However, those techniques all relied on the equations being linear and with constant coefficients. In practice, though, most interesting phenomena lead to nonlinear differential equations or linear equations with non-constant coefficients. This course focuses on solving such PDEs, both first order and second order. You will develop a number of tools and techniques, build an analysis of the underlying theory, and explore a mix of applications.

The course builds on some of the ideas from the 2nd year DEs I and DEs II courses, though it is worthwhile to note that DEs II was focussed entirely on ordinary differential equations, and while some ideas do carry over, you could probably get by if you didn't take that course.

16 B5.3 Viscous Flow

Term - Michaelmas

To follow.

17 B5.4 Waves and Compressible Flow

Term - Hilary

Propagating disturbances, or waves, occur frequently in applied mathematics. This course will be centred on some prototypical examples from fluid dynamics, the most familiar being sound waves and surface gravity waves (introduced in A10: Fluids and Waves). The models for compressible flow will be derived and then analysed for small amplitude motion, shedding light on the important phenomena of dispersion and resonance and the differences between supersonic and subsonic flow. Larger amplitude motion of liquids and gases will be described by incorporating non-linear effects, and the theory of characteristics for partial differential equations will be applied to understand the shock waves associated with supersonic flight.

18 B5.5 Further Mathematical Biology

Term - Michaelmas

To follow.

19 B5.6 Nonlinear Systems

Term - Hilary

This course will provide an introduction to the tools and concepts of dynamical systems theory which have become a central tool of both pure and applied mathematics with applications in celestial mechanics, mathematical biology, fluid dynamics, granular media, and social sciences. The course will focus on the geometry of both ordinary differential equations and maps. It will introduce important concepts of stability, bifurcations and chaos for the dynamical systems.

No special prerequisites besides the A1: Differential Equations 1 course are required. The problem sheets will require basic skills in numerical computation (numerical integration and visualisation of solutions of differential equations).

20 B6.1 Numerical Solution of Differential Equations I

Term - Michaelmas

Initial value problems are ubiquitous in mathematical models. Unfortunately, in most cases it is not possible to compute their solution analytically. However, one can use numerical methods to approximate these solutions with sufficient accuracy.

In this course you will study numerical methods for initial value problems. In particular, you will learn how to formulate, analyse, and implement these methods.

This course builds on ideas from the prelims courses on analysis and calculus. Having some knowledge of Matlab and numerical analysis (as presented in A7) can be advantageous (but is not essential).

21 B6.1 Numerical Solution of Differential Equations II

Term - Hilary

Once you know that the solutions of your partial differential equations exist and they are unique, how do we actually compute approximate solutions? What properties do these approximations possess? For instance, do they inherit the behaviour of the original solutions to the set of equations?

The course discusses both theoretical and practical aspects of discretisation techniques. We will establish stability and convergence, derive error bounds, and address the implementation of finite difference and finite volume schemes for elliptic PDEs and hyperbolic conservation laws.

22 B6.3 Integer Programming

Term - Michaelmas

Optimisation or “Mathematical Programming” is the branch of mathematics that deals with the optimal allocation of scarce resources in planning problems. An optimisation model consists of an objective, which is to be maximised or minimised and is expressed as a function of decision variables, and constraints on the decision variables expressed as functional inequalities or equalities. Constraints can be seen as modelling trade-offs under which the value of one decision variable restricts the choice of values of the other variables.

Optimisation models appear in a huge number of applications of mathematics to real world problems, ranging from the efficient planning of supply chains and logistic networks to telecommunications, medical imaging, shape optimisation in engineering, likelihood maximisation in statistical estimation, machine learning and many other domains. In this course we focus on problems that can be cast in terms of a linear objective function, linear equality and inequality constraints, and additionally, constraints that force some of the variables to take integer values. Such problems are generally hard to solve computationally, but we will learn how to break them down into a sequence of sub problems that are much easier to solve, both based on mathematical insights and through the design of clever algorithms.

23 B7.1 Classical Mechanics

Term - Michaelmas

Conserved quantities such as the energy and angular momentum of planetary orbits and rigid bodies give both physical intuition for the behaviour of such systems and tools for solving their underlying differential equations. They turn out to have a deep duality with the classical symmetries of these systems enshrined in Emmy Noether's theorem. In order to arrive at this theorem, we have to change our foundations of physical laws. In classical mechanics, the differential equations of physical systems are no longer understood as consequences of Newton's laws, but via Hamilton's principle of least action. This leads to new perspectives on physical systems generalising the phase-plane to phase-space and understanding solutions there in terms of generating functions.

This course gives a rich toolkit for analysing and solving the equations of dynamical systems arising across a variety of applications from mathematical physics to engineering and mechanical modelling. It links directly to the classical limit of quantum phenomena and provides a more flexible and sophisticated starting point for quantization.

The course builds directly on the prelims dynamics course and links on the one hand to the quantum theory courses, providing also a general context for the later mathematical physics courses, and on the other to the calculus of variations, manifolds and nonlinear systems.

24 B7.2 Electromagnetism

Term - Hilary

To follow.

25 B7.3 Further Quantum Theory

Term - Hilary

Quantum theory stands as one of the twin pillars of modern (twentieth century and later) theoretical physics, and the probabilistic nature of a "quantum" world represents a radical departure from the classical physics of Newton and Maxwell. As a framework, quantum theory forms the backbone of contemporary theories of particle physics and quantum gravity, while also underpinning important real-world technologies like lasers and microelectronics, as well as next-generation technologies like quantum computers.

In Part A Quantum Theory, you will have met the main actors of non-relativistic quantum mechanics: wave functions, the Schrödinger equation, the linearity of state-space (Hilbert space), and the Heisenberg uncertainty principle. You will have also seen how this cast

of characters comes together in a number of important examples, such as the harmonic oscillator and the Hydrogen atom.

In this Part B course you will both broaden and deepen your understanding of this profound subject. We will pursue in greater depth the abstract formalism underlying quantum theory, with a particular emphasis on the treatment and consequences of symmetries (such as rotational symmetry). We will develop tools that allow us to study substantially more complicated systems than those of Part A. These will come in the form of a variety of approximation schemes: Rayleigh-Schrödinger perturbation theory, the WKB approximation, variational methods, and Dirac-Feynman time-dependent perturbation theory. By studying the systems of multiple quantum mechanical particles, we will encounter the notion of bosons and fermions, and we will see how basic properties of the elementary particles can go a long way towards explaining the structure of matter through the periodic table of elements.

Throughout the course, we will observe deep connections between quantum theory and a diverse array of disciplines in pure and applied mathematics. In addition to the prominent role (familiar from Part A) played by certain partial differential equations, we will also confront topics in functional analysis and the theory of Lie groups and their representations. Though beyond the scope of the course, ideas from quantum mechanics have also played important roles in modern developments in geometry and topology. No previous study of these subjects is assumed.

26 B8.1 Probability, Measure and Martingales

Term - Michaelmas

In the last fifty years probability theory has emerged both as a core mathematical discipline, sitting alongside geometry, algebra and analysis, and as a fundamental way of thinking about the world. It provides the rigorous mathematical framework necessary for modelling and understanding the inherent randomness in the world around us. It has become an indispensable tool in many disciplines - from physics to neuroscience, from genetics to communication networks, and, of course, in mathematical finance. Equally, probabilistic approaches have gained importance in mathematics itself, from number theory to partial differential equations.

Our aim in this course is to introduce some of the key tools that allow us to unlock this mathematical framework. We build on the measure theory that we learned in Part A Integration and develop the mathematical foundations essential for more advanced courses in analysis and probability. We then introduce the powerful concept of martingales and explore just a few of their remarkable properties.

27 B8.2 Continuous Martingales and Stochastic Calculus

Term - Hilary

Many models of the real world, from physics, biology, finance and other areas, involve random phenomena. At the same time, we know that calculus is a key tool in our mathematical understanding of many problems. How can we put these two ideas together? In this course, we look at why this is not an easy task, due to the properties of the most fundamental continuous random process - Brownian motion. We will then look at how we can build a theory of 'stochastic' calculus, for a wide class of random processes. This leads to a new chain rule (given by Itô's lemma), and new notions of convergence for the integral.

This course builds on the Part A Integration and Metric Spaces courses, and more directly from the (Michaelmas Term) Part B Probability, Measure and Martingales course. It complements the Part B Mathematical Models of Financial Derivatives and Applied Probability courses, and leads on to the Part C Stochastic Differential Equations and Stochastic Analysis and PDEs courses.

28 B8.3 Mathematical Models of Financial Derivatives

Term - Hilary

To follow.

29 B8.4 Information Theory

Term - Michaelmas

Information theory was introduced by C. E. Shannon in the late 1940's under the name "Communication theory". Its purpose is to provide a mathematical formalisation of communication as a transmission of a piece information over a channel. This leads to the following fundamental questions:

- How do we measure the information content of a message or a signal?
- How to compress and store data?
- How can we efficiently transmit information over a noisy channel and what are the limitations?

This theory is now widely used in applications, in fact you are constantly using in your everyday life! Live streaming, music and image compression (mp3 and jpg), phone calls, data storage ...

In this course we will introduce the fundamental concepts of entropy, mutual information and divergence, and how they relate to information transmission. These are then used to analyse source coding (efficient storage of data) and channel coding (coping with transmission noise).

Concerning prerequisites, Part A Probability would be very helpful, but is not essential.

30 B8.5 Graph Theory

Term - Michaelmas

To follow.

31 BO1.1 History of Mathematics

Term - Michaelmas and Hilary

When studying mathematics, we encounter many names attached to theorems: Euler, Gauss, Cauchy, Cayley, ... But who were these people? And what motivated them to develop the mathematics that they did? Indeed, where do they fit within the overall picture of the history of mathematics? And how did the mathematics that we study arise in the first place?

The advantage of looking at earlier stages in the development of mathematics is that it rids us of the prejudice that mathematics must always have appeared as it does now – on the contrary, each culture and period has its own version of mathematics, which must be understood on its own terms. The study of the history of mathematics not only enhances the understanding of mathematics, but also provides an opportunity to explore other cultures and points of view.

The purpose of this course is to look in detail at the history of the mathematics that you have studied during the core courses of your first two years. We will consider not only the mathematics, but also the people, and the historical contexts within which they were working. A natural follow-up to this course at Part C is a COD Dissertation on the History of Mathematics.

32 BN1 Mathematics Education

Term - Michaelmas and Hilary

We have been doing it all of our lives, but what does it mean to learn mathematics? As we begin to address this question, we go on to ask what it means to teach mathematics. What is the teacher's role in students' learning of mathematics? We recognise that teaching and learning happen in different cultural and institutional contexts, so how do these contexts shape the experiences of what it means to learn mathematics for individuals and groups?

These questions motivate the design of the BN1 course in Mathematics Education. The course is in two parts: BN1.1 is a University-based lecture course in Michaelmas Term, BN1.2 is largely a school-based experience in Hilary Term with additional University-based sessions. In BN1.1 we draw on your own experiences of learning mathematics, and often your experiences at Oxford. In BN1.2 when in school, the focus is more on young people's experiences of learning mathematics, with a particular emphasis on A level students: you will collect data about their understanding of Calculus and design a session for them on an aspect of degree-level mathematics. Typically you will choose to follow both parts of the course, but it is possible just to take the BN1.1 course as a stand-alone option. BN1.2 is not available as a stand-alone option.

The course as a whole is well suited to those who have ambitions to teach mathematics, in whatever context that might be. It is also suitable for anyone hoping for a deeper understanding of mathematics education, but the BN1.2 element is only available to those who can play an ambassadorial role for the University and the subject within local state schools and for this reason we hold interviews at the start of Michaelmas Term.