B4.1 Functional Analysis I MT 2018: Problem Sheet 3

1. Let f be a real-valued continuous function on $[0,1]$ such that

$$\int_0^1 f(t)\mathrm{e}^{nt}\mathrm{d}t = 0 \quad \text{for } n = 0, 1, 2, \ldots .$$

Prove that $f \equiv 0$.

2. Consider the space $c_0 = \{x \in \ell^\infty(\mathbb{R}) : \lim_{j\to\infty} x_j = 0\}$. Make use of the Weierstrass polynomial approximation theorem to prove that, given $\varepsilon > 0$, and any $x \in c_0$ there exists a natural number N and scalars $\lambda_1, \ldots, \lambda_N$ such that

$$\left| x_j - \sum_{n=1}^{N} \lambda_n j^{-n} \right| < \varepsilon \qquad (j = 1, 2, \ldots).$$

3. *Throughout this question take care to make it clear which norm you are working with when more than one norm is in play.*

(i) Let $a \leqslant c < d \leqslant b$. Prove that there exists $f_n \in C[a,b]$ such that $f_n \to \chi_{[c,d]}$ in the sense of L^1, i.e. $\|f_n - \chi_{[c,d]}\|_{L^1} \to 0$.
Using that the set of step functions $L^{\text{step}}[a,b]$ is dense in $(L^1[a,b], \|\cdot\|_1)$ deduce that $C[a,b]$ is dense in $(L^1[a,b], \|\cdot\|_1)$.

(ii) Let $(X, \|\cdot\|_X)$ be a normed space, let $Z \subset Y \subset X$ subspaces and let $\|\cdot\|_Y$ be a norm on Y. Suppose that Z is dense in $(Y, \|\cdot\|_Y)$ and that Y is dense in $(X, \|\cdot\|_X)$. Prove that if there exists some $C \in \mathbb{R}$ so that

$$(\star) \qquad \|y\|_X \leqslant C\|y\|_Y \text{ for all } y \in Y$$

then also Z is dense in $(X, \|\cdot\|_X)$.
Conversely, illustrate by an example that this claim does not hold true in general if the assumption $(\star)$ is dropped.

4. (i) Let $(X, \|\cdot\|_{Lip})$ be the space of Lipschitz functions on $[-1,1]$ with $x(0) = 0$, as defined in Q. 4 on Problem sheet 1. By considering the family $\{\, x_u \mid u \in [-1,1]\,\}$, where, for $t \in [-1,1]$,

$$x_u(t) = |u| - |u - t|,$$

or otherwise, prove that X is inseparable.

(ii) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and suppose that there exists a linear map $T : X \to Y$ which is isometric (though not necessarily surjective). Prove that if X is inseparable then also Y must be inseparable.
Does the converse implication, i.e. that Y inseparable implies X inseparable, hold? (Please justify your answer).

(ii) Let $C^b([1,\infty))$ be the space of bounded complex-valued continuous functions on $[1,\infty)$ with the sup norm.
By constructing a suitable map T from ℓ^∞ into $C^b([1,\infty))$, or otherwise, prove that $C^b([1,\infty))$ is inseparable.

5. (i) Let c be the subspace of ℓ^∞ consisting of all convergent sequences. Prove that c is separable.

(ii) Let K be the set

$$K = \{\, z \in \mathbb{C} \mid 1 \leqslant |z| \leqslant 2 \,\}.$$

Define $\varphi\colon C(K) \to \mathbb{C}$ by

$$\varphi(f) = \int_{|z|=3/2} f(z)\,\mathrm{d}z.$$

Prove that φ is a continuous linear functional on $C(K)$. Use this to prove that the space of all complex polynomials is not dense in $C(K, \mathbb{C})$. *Hint: consider the function* $f(z) = 1/z \in C(K, \mathbb{C})$.